Grades Eight Through Twelve

Introduction

The standards for grades eight through twelve are organized differently from those for kindergarten through grade seven. In this section strands are not used for organizational purposes as they are in the elementary grades because the mathematics studied in grades eight through twelve falls naturally under discipline headings: algebra, geometry, and so forth. Many schools teach this material in traditional courses; others teach it in an integrated fashion. To allow local educational agencies and teachers flexibility in teaching the material, the standards for grades eight through twelve do not mandate that a particular discipline be initiated and completed in a single grade. The core content of these subjects must be covered; students are expected to achieve the standards however these subjects are sequenced.

Standards are provided for algebra I, geometry, algebra II, trigonometry, mathematical analysis, linear algebra, probability and statistics, Advanced Placement probability and statistics, and calculus. Many of the more advanced subjects are not taught in every middle school or high school. Moreover, schools and districts have different ways of combining the subject matter in these various disciplines. For example, many schools combine some trigonometry, mathematical analysis, and linear algebra to form a precalculus course. Some districts prefer offering trigonometry content with algebra II.

Table 1, "Mathematics Disciplines, by Grade Level," reflects typical grade-level groupings of these disciplines in both integrated and traditional curricula. The lightly shaded region reflects the minimum requirement for mastery by all students. The dark shaded region depicts content that is typically considered elective but that should also be mastered by students who complete the other disciplines in the lower grade levels and continue the study of mathematics.

Table 1

Mathematics Disciplines, by Grade Level

	Grades				
Discipline	*Eight*	*Nine*	*Ten*	*Eleven*	*Twelve*
Algebra I					
Geometry					
Algebra II					
Probability and Statistics					
Trigonometry					
Linear Algebra					
Mathematical Analysis					
Advanced Placement Probability and Statistics					
Calculus					

Many other combinations of these advanced subjects into courses are possible. What is described in this section are standards for the academic content by discipline; this document does not endorse a particular choice of structure for courses or a particular method of teaching the mathematical content.

When students delve deeply into mathematics, they gain not only conceptual understanding of mathematical principles but also knowledge of, and experience with, pure reasoning. One of the most important goals of mathematics is to teach students logical reasoning. The logical reasoning inherent in the study of mathematics allows for applications to a broad range of situations in which answers to practical problems can be found with accuracy.

By grade eight, students' mathematical sensitivity should be sharpened. Students need to start perceiving logical subtleties and appreciate the need for sound mathematical arguments before making conclusions. As students progress in the study of mathematics, they learn to distinguish between inductive and deductive reasoning; understand the meaning of logical implication; test general assertions; realize that one counterexample is enough to show that a general assertion is false; understand conceptually that although a general assertion is true in a few cases, it is not true in all cases; distinguish between something being proven and a mere plausibility argument; and identify logical errors in chains of reasoning.

Mathematical reasoning and conceptual understanding are not separate from content; they are intrinsic to the mathematical discipline students master at more advanced levels.

Algebra I

Symbolic reasoning and calculations with symbols are central in algebra. Through the study of algebra, a student develops an understanding of the symbolic language of mathematics and the sciences. In addition, algebraic skills and concepts are developed and used in a wide variety of problem-solving situations.

1.0 Students identify and use the arithmetic properties of subsets of integers and rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable:

1.1 Students use properties of numbers to demonstrate whether assertions are true or false.

2.0 Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

3.0 Students solve equations and inequalities involving absolute values.

4.0 Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.

5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

6.0 Students graph a linear equation and compute the x- and y-intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

7.0 Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.

8.0 Students understand the concepts of parallel lines and perpendicular lines and how those slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

9.0 Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

10.0 Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

11.0 Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

12.0 Students simplify fractions with polynomials in the numerator and denominator by factoring both and reducing them to the lowest terms.

13.0 Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.

14.0 Students solve a quadratic equation by factoring or completing the square.

15.0 Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

16.0 Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.

17.0 Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.

18.0 Students determine whether a relation defined by a graph, a set of ordered pairs, or a symbolic expression is a function and justify the conclusion.

19.0 Students know the quadratic formula and are familiar with its proof by completing the square.

20.0 Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.

21.0 Students graph quadratic functions and know that their roots are the x-intercepts.

22.0 Students use the quadratic formula or factoring techniques or both to determine whether the graph of a quadratic function will intersect the x-axis in zero, one, or two points.

23.0 Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.

24.0 Students use and know simple aspects of a logical argument:

24.1 Students explain the difference between inductive and deductive reasoning and identify and provide examples of each.

24.2 Students identify the hypothesis and conclusion in logical deduction.

24.3 Students use counterexamples to show that an assertion is false and recognize that a single counterexample is sufficient to refute an assertion.

25.0 Students use properties of the number system to judge the validity of results, to justify each step of a procedure, and to prove or disprove statements:

25.1 Students use properties of numbers to construct simple, valid arguments (direct and indirect) for, or formulate counterexamples to, claimed assertions.

25.2 Students judge the validity of an argument according to whether the properties of the real number system and the order of operations have been applied correctly at each step.

25.3 Given a specific algebraic statement involving linear, quadratic, or absolute value expressions or equations or inequalities, students determine whether the statement is true sometimes, always, or never.

Geometry

The geometry skills and concepts developed in this discipline are useful to all students. Aside from learning these skills and concepts, students will develop their ability to construct formal, logical arguments and proofs in geometric settings and problems.

1.0 Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning.

2.0 Students write geometric proofs, including proofs by contradiction.

3.0 Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

4.0 Students prove basic theorems involving congruence and similarity.

5.0 Students prove that triangles are congruent or similar, and they are able to use the concept of corresponding parts of congruent triangles.

6.0 Students know and are able to use the triangle inequality theorem.

7.0 Students prove and use theorems involving the properties of parallel lines cut by a transversal, the properties of quadrilaterals, and the properties of circles.

8.0 Students know, derive, and solve problems involving the perimeter, circumference, area, volume, lateral area, and surface area of common geometric figures.

9.0 Students compute the volumes and surface areas of prisms, pyramids, cylinders, cones, and spheres; and students commit to memory the formulas for prisms, pyramids, and cylinders.

10.0 Students compute areas of polygons, including rectangles, scalene triangles, equilateral triangles, rhombi, parallelograms, and trapezoids.

11.0 Students determine how changes in dimensions affect the perimeter, area, and volume of common geometric figures and solids.

12.0 Students find and use measures of sides and of interior and exterior angles of triangles and polygons to classify figures and solve problems.

13.0 Students prove relationships between angles in polygons by using properties of complementary, supplementary, vertical, and exterior angles.

14.0 Students prove the Pythagorean theorem.

15.0 Students use the Pythagorean theorem to determine distance and find missing lengths of sides of right triangles.

16.0 Students perform basic constructions with a straightedge and compass, such as angle bisectors, perpendicular bisectors, and the line parallel to a given line through a point off the line.

17.0 Students prove theorems by using coordinate geometry, including the midpoint of a line segment, the distance formula, and various forms of equations of lines and circles.

18.0 Students know the definitions of the basic trigonometric functions defined by the angles of a right triangle. They also know and are able to use elementary relationships between them. For example, $\tan(x) = \sin(x)/\cos(x)$, $(\sin(x))^2 + (\cos(x))^2 = 1$.

19.0 Students use trigonometric functions to solve for an unknown length of a side of a right triangle, given an angle and a length of a side.

20.0 Students know and are able to use angle and side relationships in problems with special right triangles, such as 30°, 60°, and 90° triangles and 45°, 45°, and 90° triangles.

21.0 Students prove and solve problems regarding relationships among chords, secants, tangents, inscribed angles, and inscribed and circumscribed polygons of circles.

22.0 Students know the effect of rigid motions on figures in the coordinate plane and space, including rotations, translations, and reflections.

Algebra II

This discipline complements and expands the mathematical content and concepts of algebra I and geometry. Students who master algebra II will gain experience with algebraic solutions of problems in various content areas, including the solution of systems of quadratic equations, logarithmic and exponential functions, the binomial theorem, and the complex number system.

1.0 Students solve equations and inequalities involving absolute value.

2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or with matrices.

3.0 Students are adept at operations on polynomials, including long division.

4.0 Students factor polynomials representing the difference of squares, perfect square trinomials, and the sum and difference of two cubes.

5.0 Students demonstrate knowledge of how real and complex numbers are related both arithmetically and graphically. In particular, they can plot complex numbers as points in the plane.

6.0 Students add, subtract, multiply, and divide complex numbers.

7.0 Students add, subtract, multiply, divide, reduce, and evaluate rational expressions with monomial and polynomial denominators and simplify complicated rational expressions, including those with negative exponents in the denominator.

8.0 Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.

9.0 Students demonstrate and explain the effect that changing a coefficient has on the graph of quadratic functions; that is, students can determine how the graph of a parabola changes as a, b, and c vary in the equation $y = a(x\text{-}b)^2 + c$.

10.0 Students graph quadratic functions and determine the maxima, minima, and zeros of the function.

11.0 Students prove simple laws of logarithms.

11.1 Students understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

11.2 Students judge the validity of an argument according to whether the properties of real numbers, exponents, and logarithms have been applied correctly at each step.

12.0 Students know the laws of fractional exponents, understand exponential functions, and use these functions in problems involving exponential growth and decay.

13.0 Students use the definition of logarithms to translate between logarithms in any base.

14.0 Students understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values.

15.0 Students determine whether a specific algebraic statement involving rational expressions, radical expressions, or logarithmic or exponential functions is sometimes true, always true, or never true.

16.0 Students demonstrate and explain how the geometry of the graph of a conic section (e.g., asymptotes, foci, eccentricity) depends on the coefficients of the quadratic equation representing it.

17.0 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, students can use the method for completing the square to put the equation into standard form and can recognize whether the graph of the equation is a circle, ellipse, parabola, or hyperbola. Students can then graph the equation.

18.0 Students use fundamental counting principles to compute combinations and permutations.

19.0 Students use combinations and permutations to compute probabilities.

20.0 Students know the binomial theorem and use it to expand binomial expressions that are raised to positive integer powers.

21.0 Students apply the method of mathematical induction to prove general statements about the positive integers.

22.0 Students find the general term and the sums of arithmetic series and of both finite and infinite geometric series.

23.0 Students derive the summation formulas for arithmetic series and for both finite and infinite geometric series.

24.0 Students solve problems involving functional concepts, such as composition, defining the inverse function and performing arithmetic operations on functions.

25.0 Students use properties from number systems to justify steps in combining and simplifying functions.

Trigonometry

Trigonometry uses the techniques that students have previously learned from the study of algebra and geometry. The trigonometric functions studied are defined geometrically rather than in terms of algebraic equations. Facility with these functions as well as the ability to prove basic identities regarding them is especially important for students intending to study calculus, more advanced mathematics, physics and other sciences, and engineering in college.

1.0 Students understand the notion of angle and how to measure it, in both degrees and radians. They can convert between degrees and radians.

2.0 Students know the definition of sine and cosine as y- and x-coordinates of points on the unit circle and are familiar with the graphs of the sine and cosine functions.

3.0 Students know the identity $\cos^2(x) + \sin^2(x) = 1$:

3.1 Students prove that this identity is equivalent to the Pythagorean theorem (i.e., students can prove this identity by using the Pythagorean theorem and, conversely, they can prove the Pythagorean theorem as a consequence of this identity).

3.2 Students prove other trigonometric identities and simplify others by using the identity $\cos^2(x) + \sin^2(x) = 1$. For example, students use this identity to prove that $\sec^2(x) = \tan^2(x) + 1$.

4.0 Students graph functions of the form $f(t) = A \sin(Bt + C)$ or $f(t) = A \cos(Bt + C)$ and interpret A, B, and C in terms of amplitude, frequency, period, and phase shift.

5.0 Students know the definitions of the tangent and cotangent functions and can graph them.

6.0 Students know the definitions of the secant and cosecant functions and can graph them.

7.0 Students know that the tangent of the angle that a line makes with the x-axis is equal to the slope of the line.

8.0 Students know the definitions of the inverse trigonometric functions and can graph the functions.

9.0 Students compute, by hand, the values of the trigonometric functions and the inverse trigonometric functions at various standard points.

10.0 Students demonstrate an understanding of the addition formulas for sines and cosines and their proofs and can use those formulas to prove and/or simplify other trigonometric identities.

11.0 Students demonstrate an understanding of half-angle and double-angle formulas for sines and cosines and can use those formulas to prove and/or simplify other trigonometric identities.

12.0 Students use trigonometry to determine unknown sides or angles in right triangles.

13.0 Students know the law of sines and the law of cosines and apply those laws to solve problems.

14.0 Students determine the area of a triangle, given one angle and the two adjacent sides.

15.0 Students are familiar with polar coordinates. In particular, they can determine polar coordinates of a point given in rectangular coordinates and vice versa.

16.0 Students represent equations given in rectangular coordinates in terms of polar coordinates.

17.0 Students are familiar with complex numbers. They can represent a complex number in polar form and know how to multiply complex numbers in their polar form.

18.0 Students know DeMoivre's theorem and can give nth roots of a complex number given in polar form.

19.0 Students are adept at using trigonometry in a variety of applications and word problems.

Mathematical Analysis

This discipline combines many of the trigonometric, geometric, and algebraic techniques needed to prepare students for the study of calculus and strengthens their conceptual understanding of problems and mathematical reasoning in solving problems. These standards take a functional point of view toward those topics. The most significant new concept is that of limits. Mathematical analysis is often combined with a course in trigonometry or perhaps with one in linear algebra to make a year-long precalculus course.

1.0 Students are familiar with, and can apply, polar coordinates and vectors in the plane. In particular, they can translate between polar and rectangular coordinates and can interpret polar coordinates and vectors graphically.

2.0 Students are adept at the arithmetic of complex numbers. They can use the trigonometric form of complex numbers and understand that a function of a complex variable can be viewed as a function of two real variables. They know the proof of DeMoivre's theorem.

3.0 Students can give proofs of various formulas by using the technique of mathematical induction.

4.0 Students know the statement of, and can apply, the fundamental theorem of algebra.

5.0 Students are familiar with conic sections, both analytically and geometrically:

5.1 Students can take a quadratic equation in two variables; put it in standard form by completing the square and using rotations and translations, if necessary; determine what type of conic section the equation represents; and determine its geometric components (foci, asymptotes, and so forth).

5.2 Students can take a geometric description of a conic section—for example, the locus of points whose sum of its distances from (1, 0) and (-1, 0) is 6—and derive a quadratic equation representing it.

6.0 Students find the roots and poles of a rational function and can graph the function and locate its asymptotes.

7.0 Students demonstrate an understanding of functions and equations defined parametrically and can graph them.

8.0 Students are familiar with the notion of the limit of a sequence and the limit of a function as the independent variable approaches a number or infinity. They determine whether certain sequences converge or diverge.

Linear Algebra

The general goal in this discipline is for students to learn the techniques of matrix manipulation so that they can solve systems of linear equations in any number of variables. Linear algebra is most often combined with another subject, such as trigonometry, mathematical analysis, or precalculus.

1.0 Students solve linear equations in any number of variables by using Gauss-Jordan elimination.

2.0 Students interpret linear systems as coefficient matrices and the Gauss-Jordan method as row operations on the coefficient matrix.

3.0 Students reduce rectangular matrices to row echelon form.

4.0 Students perform addition on matrices and vectors.

5.0 Students perform matrix multiplication and multiply vectors by matrices and by scalars.

6.0 Students demonstrate an understanding that linear systems are inconsistent (have no solutions), have exactly one solution, or have infinitely many solutions.

7.0 Students demonstrate an understanding of the geometric interpretation of vectors and vector addition (by means of parallelograms) in the plane and in three-dimensional space.

8.0 Students interpret geometrically the solution sets of systems of equations. For example, the solution set of a single linear equation in two variables is interpreted as a line in the plane, and the solution set of a two-by-two system is interpreted as the intersection of a pair of lines in the plane.

9.0 Students demonstrate an understanding of the notion of the inverse to a square matrix and apply that concept to solve systems of linear equations.

10.0 Students compute the determinants of 2×2 and 3×3 matrices and are familiar with their geometric interpretations as the area and volume of the parallelepipeds spanned by the images under the matrices of the standard basis vectors in two-dimensional and three-dimensional spaces.

11.0 Students know that a square matrix is invertible if, and only if, its determinant is nonzero. They can compute the inverse to 2×2 and 3×3 matrices using row reduction methods or Cramer's rule.

12.0 Students compute the scalar (dot) product of two vectors in n-dimensional space and know that perpendicular vectors have zero dot product.

Probability and Statistics

This discipline is an introduction to the study of probability, interpretation of data, and fundamental statistical problem solving. Mastery of this academic content will provide students with a solid foundation in probability and facility in processing statistical information.

1.0 Students know the definition of the notion of *independent events* and can use the rules for addition, multiplication, and complementation to solve for probabilities of particular events in finite sample spaces.

2.0 Students know the definition of *conditional probability* and use it to solve for probabilities in finite sample spaces.

3.0 Students demonstrate an understanding of the notion of *discrete random variables* by using them to solve for the probabilities of outcomes, such as the probability of the occurrence of five heads in 14 coin tosses.

4.0 Students are familiar with the standard distributions (normal, binomial, and exponential) and can use them to solve for events in problems in which the distribution belongs to those families.

5.0 Students determine the mean and the standard deviation of a normally distributed random variable.

6.0 Students know the definitions of the *mean, median,* and *mode* of a distribution of data and can compute each in particular situations.

7.0 Students compute the variance and the standard deviation of a distribution of data.

8.0 Students organize and describe distributions of data by using a number of different methods, including frequency tables, histograms, standard line and bar graphs, stem-and-leaf displays, scatterplots, and box-and-whisker plots.

Advanced Placement Probability and Statistics

This discipline is a technical and in-depth extension of probability and statistics. In particular, mastery of academic content for advanced placement gives students the background to succeed in the Advanced Placement examination in the subject.

1.0 Students solve probability problems with finite sample spaces by using the rules for addition, multiplication, and complementation for probability distributions and understand the simplifications that arise with independent events.

2.0 Students know the definition of *conditional probability* and use it to solve for probabilities in finite sample spaces.

3.0 Students demonstrate an understanding of the notion of *discrete random variables* by using this concept to solve for the probabilities of outcomes, such as the probability of the occurrence of five or fewer heads in 14 coin tosses.

4.0 Students understand the notion of a *continuous random variable* and can interpret the probability of an outcome as the area of a region under the graph of the probability density function associated with the random variable.

5.0 Students know the definition of the *mean of a discrete random variable* and can determine the mean for a particular discrete random variable.

6.0 Students know the definition of the *variance of a discrete random variable* and can determine the variance for a particular discrete random variable.

7.0 Students demonstrate an understanding of the standard distributions (normal, binomial, and exponential) and can use the distributions to solve for events in problems in which the distribution belongs to thoe families.

8.0 Students determine the mean and the standard deviation of a normally distributed random variable.

9.0 Students know the central limit theorem and can use it to obtain approximations for probabilities in problems of finite sample spaces in which the probabilities are distributed binomially.

10.0 Students know the definitions of the *mean*, *median*, and *mode of distribution* of data and can compute each of them in particular situations.

11.0 Students compute the variance and the standard deviation of a distribution of data.

12.0 Students find the line of best fit to a given distribution of data by using least squares regression.

13.0 Students know what the *correlation coefficient of two variables* means and are familiar with the coefficient's properties.

14.0 Students organize and describe distributions of data by using a number of different methods, including frequency tables, histograms, standard line graphs and bar graphs, stem-and-leaf displays, scatterplots, and box-and-whisker plots.

15.0 Students are familiar with the notions of a statistic of a distribution of values, of the sampling distribution of a statistic, and of the variability of a statistic.

16.0 Students know basic facts concerning the relation between the mean and the standard deviation of a sampling distribution and the mean and the standard deviation of the population distribution.

17.0 Students determine confidence intervals for a simple random sample from a normal distribution of data and determine the sample size required for a desired margin of error.

18.0 Students determine the P-value for a statistic for a simple random sample from a normal distribution.

19.0 Students are familiar with the *chi*-square distribution and *chi*-square test and understand their uses.

Calculus

When taught in high school, calculus should be presented with the same level of depth and rigor as are entry-level college and university calculus courses. These standards outline a complete college curriculum in one variable calculus. Many high school programs may have insufficient time to cover all of the following content in a typical academic year. For example, some districts may treat differential equations lightly and spend substantial time on infinite sequences and series. Others may do the opposite. Consideration of the College Board syllabi for the Calculus AB and Calculus BC sections of the Advanced Placement Examination in Mathematics may be helpful in making curricular decisions. Calculus is a widely applied area of mathematics and involves a beautiful intrinsic theory. Students mastering this content will be exposed to both aspects of the subject.

1.0 Students demonstrate knowledge of both the formal definition and the graphical interpretation of limit of values of functions. This knowledge includes one-sided limits, infinite limits, and limits at infinity. Students know the definition of convergence and divergence of a function as the domain variable approaches either a number or infinity:

- 1.1 Students prove and use theorems evaluating the limits of sums, products, quotients, and composition of functions.
- 1.2 Students use graphical calculators to verify and estimate limits.
- 1.3 Students prove and use special limits, such as the limits of $(\sin(x))/x$ and $(1\text{-}\cos(x))/x$ as x tends to 0.

2.0 Students demonstrate knowledge of both the formal definition and the graphical interpretation of continuity of a function.

3.0 Students demonstrate an understanding and the application of the intermediate value theorem and the extreme value theorem.

4.0 Students demonstrate an understanding of the formal definition of the derivative of a function at a point and the notion of differentiability:

- 4.1 Students demonstrate an understanding of the derivative of a function as the slope of the tangent line to the graph of the function.

4.2 Students demonstrate an understanding of the interpretation of the derivative as an instantaneous rate of change. Students can use derivatives to solve a variety of problems from physics, chemistry, economics, and so forth that involve the rate of change of a function.

4.3 Students understand the relation between differentiability and continuity.

4.4 Students derive derivative formulas and use them to find the derivatives of algebraic, trigonometric, inverse trigonometric, exponential, and logarithmic functions.

5.0 Students know the chain rule and its proof and applications to the calculation of the derivative of a variety of composite functions.

6.0 Students find the derivatives of parametrically defined functions and use implicit differentiation in a wide variety of problems in physics, chemistry, economics, and so forth.

7.0 Students compute derivatives of higher orders.

8.0 Students know and can apply Rolle's theorem, the mean value theorem, and L'Hôpital's rule.

9.0 Students use differentiation to sketch, by hand, graphs of functions. They can identify maxima, minima, inflection points, and intervals in which the function is increasing and decreasing.

10.0 Students know Newton's method for approximating the zeros of a function.

11.0 Students use differentiation to solve optimization (maximum-minimum problems) in a variety of pure and applied contexts.

12.0 Students use differentiation to solve related rate problems in a variety of pure and applied contexts.

13.0 Students know the definition of the definite integral by using Riemann sums. They use this definition to approximate integrals.

14.0 Students apply the definition of the integral to model problems in physics, economics, and so forth, obtaining results in terms of integrals.

15.0 Students demonstrate knowledge and proof of the fundamental theorem of calculus and use it to interpret integrals as antiderivatives.

16.0 Students use definite integrals in problems involving area, velocity, acceleration, volume of a solid, area of a surface of revolution, length of a curve, and work.

17.0 Students compute, by hand, the integrals of a wide variety of functions by using techniques of integration, such as substitution, integration by parts, and trigonometric substitution. They can also combine these techniques when appropriate.

18.0 Students know the definitions and properties of inverse trigonometric functions and the expression of these functions as indefinite integrals.

19.0 Students compute, by hand, the integrals of rational functions by combining the techniques in standard 17.0 with the algebraic techniques of partial fractions and completing the square.

20.0 Students compute the integrals of trigonometric functions by using the techniques noted above.

21.0 Students understand the algorithms involved in Simpson's rule and Newton's method. They use calculators or computers or both to approximate integrals numerically.

22.0 Students understand improper integrals as limits of definite integrals.

23.0 Students demonstrate an understanding of the definitions of convergence and divergence of sequences and series of real numbers. By using such tests as the comparison test, ratio test, and alternate series test, they can determine whether a series converges.

24.0 Students understand and can compute the radius (interval) of the convergence of power series.

25.0 Students differentiate and integrate the terms of a power series in order to form new series from known ones.

26.0 Students calculate Taylor polynomials and Taylor series of basic functions, including the remainder term.

27.0 Students know the techniques of solution of selected elementary differential equations and their applications to a wide variety of situations, including growth-and-decay problems.

Glossary

absolute value. A number's distance from zero on the number line. The absolute value of -4 is 4; the absolute value of 4 is 4.

algorithm. An organized procedure for performing a given type of calculation or solving a given type of problem. An example is long division.

arithmetic sequence. A sequence of elements, $a_1, a_2, a_3, \ldots$, such that the difference of successive terms is a constant $a_{i+1} - a_i = k$; for example, the sequence {2, 5, 8, 11, 14, . . .} where the common difference is 3.

asymptotes. Straight lines that have the property of becoming and staying arbitrarily close to the curve as the distance from the origin increases to infinity. For example, the x-axis is the only asymptote to the graph of $\sin(x)/x$.

axiom. A basic assumption about a mathematical system from which theorems can be deduced. For example, the system could be the points and lines in the plane. Then an axiom would be that given any two distinct points in the plane, there is a unique line through them.

binomial. In algebra, an expression consisting of the sum or difference of two monomials (see the definition of *monomial*), such as $4a–8b$.

binomial distribution. In probability, a binomial distribution gives the probabilities of k outcomes A (or $n–k$ outcomes B) in n independent trials for a two-outcome experiment in which the possible outcomes are denoted A and B.

binomial theorem. In mathematics, a theorem that specifies the complete expansion of a binomial raised to any positive integer power.

box-and-whisker plot. A graphical method for showing the median, quartiles, and extremes of data. A box plot shows where the data are spread out and where they are concentrated.

complex numbers. Numbers that have the form $a + bi$ where a and b are real numbers and i satisfies the equation $i^2 = -1$. Multiplication is denoted by $(a+bi)(c+di) = (ac–bd) + (ad+bc)i$, and addition is denoted by $(a+bi) + (c + di) = (a+c) + (b+d)i$.

congruent. Two shapes in the plane or in space are congruent if there is a rigid motion that identifies one with the other (see the definition of *rigid motion*).

conjecture. An educated guess.

coordinate system. A rule of correspondence by which two or more quantities locate points unambiguously and which satisfies the further property that points unambiguously determine the quantities; for example, the usual Cartesian coordinates x, y in the plane.

cosine. $\cos(\theta)$ is the x-coordinate of the point on the unit circle so that the ray connecting the point with the origin makes an angle of θ with the positive x-axis. When θ is an angle of a right triangle, then $\cos(\theta)$ is the ratio of the adjacent side with the hypotenuse.

dilation. In geometry, a transformation *D* of the plane or space is a dilation at a point *P* if it takes *P* to itself, preserves angles, multiplies distances from *P* by a positive real number *r*, and takes every ray through *P* onto itself. In case *P* is the origin for a Cartesian coordinate system in the plane, then the dilation *D* maps the point (x, y) to the point (rx, ry).

dimensional analysis. A method of manipulating unit measures algebraically to determine the proper units for a quantity computed algebraically. For example, velocity has units of the form length over time (e.g., meters per second [*m/sec*]), and acceleration has units of velocity over time; so it follows that acceleration has units $(m/sec)/sec = m/(sec^2)$.

expanded form. The expanded form of an algebraic expression is the *equivalent expression* without parentheses. For example, the expanded form of $(a + b)^2$ is $a^2 + 2ab + b^2$.

exponent. The power to which a number or variable is raised (the exponent may be any real number).

exponential function. A function commonly used to study growth and decay. It has the form $y = a^x$ with *a* positive.

factors. Any of two or more quantities that are multiplied together. In the expression 3.712×11.315, the factors are 3.712 and 11.315.

function. A correspondence in which values of one variable determine the values of another.

geometric sequence. A sequence in which there is a common ratio between successive terms. Each successive term of a geometric sequence is found by multiplying the preceding term by the common ratio. For example, in the sequence {1, 3, 9, 27, 81, . . .} the common ratio is 3.

histogram. A vertical block graph with no spaces between the blocks. It is used to represent frequency data in statistics.

inequality. A relationship between two quantities indicating that one is strictly *less than* or *less than or equal* to the other.

integers. The set consisting of the positive and negative whole numbers and zero; for example, {. . . -2, -1, 0, 1, 2 . . .}.

irrational number. A number that cannot be represented as an exact ratio of two integers. For example, the square root of 2 or π.

linear expression. An expression of the form $ax + b$ where *x* is variable and *a* and *b* are constants; or in more variables, an expression of the form $ax + by + c$, $ax + by + cz + d$, etc.

linear equation. An equation containing linear expressions.

logarithm. The inverse of exponentiation; for example, $a^{\log_a x} = x$.

mean. In statistics, the average obtained by dividing the sum of two or more quantities by the number of these quantities.

median. In statistics, the quantity designating the middle value in a set of numbers.

mode. In statistics, the value that occurs most frequently in a given series of numbers.

monomial. In the variables *x, y, z*, a monomial is an expression of the form $ax^my^nz^k$, in which *m, n,* and *k* are nonnegative integers and *a* is a constant (e.g., $5x^2$, $3x^2y$ or $7x^3yz^2$).

nonstandard unit. Unit of measurement expressed in terms of objects (such as paper clips, sticks of gum, shoes, etc.).

parallel. Given distinct lines in the plane that are infinite in both directions, the lines are parallel if they never meet. Two distinct lines in the coordinate plane are parallel if and only if they have the same slope.

permutation. A permutation of the set of numbers {1, 2, . . . *n*} is a reordering of these numbers.

polar coordinates. The coordinate system for the plane based on *r*θ, the distance from the origin and θ, and the angle between the positive *x*-axis and the ray from the origin to the point.

polar equation. Any relation between the polar coordinates (r, θ) of a set of points (e.g., $r = 2\cos\theta$ is the polar equation of a circle).

polynomial. In algebra, a sum of monomials; for example, $x^2 + 2xy + y^2$.

prime. A natural number *p* greater than 1 is prime if and only if the only positive integer factors of *p* are 1 and *p*. The first seven primes are 2, 3, 5, 7, 11, 13, 17.

quadratic function. A function given by a polynomial of degree 2.

random variable. A function on a probability space.

range. In statistics, the difference between the greatest and smallest values in a data set. In mathematics, the image of a function.

ratio. A comparison expressed as a fraction. For example, there is a ratio of three boys to two girls in a class (3/2, 3:2).

rational numbers. Numbers that can be expressed as the quotient of two integers; for example, 7/3, 5/11, -5/13, 7 = 7/1.

real numbers. All rational and irrational numbers.

reflection. The reflection through a line in the plane or a plane in space is the transformation that takes each point in the plane to its mirror image with respect to the line or its mirror image with respect to the plane in space. It produces a mirror image of a geometric figure.

rigid motion. A transformation of the plane or space, which preserves distance and angles.

root extraction. Finding a number that can be used as a factor a given number of times to produce the original number; for example, the fifth root of 32 = 2 because $2 \times 2 \times 2 \times 2 \times 2 = 32$).

rotation. A rotation in the plane through an angle θ and about a point *P* is a rigid motion *T* fixing *P* so that if *Q* is distinct from *P*, then the angle between the lines *PQ* and *PT(Q)* is always θ. A rotation through an angle θ in space is a rigid motion *T* fixing the points of a line *l* so that it is a rotation through θ in the plane perpendicular to *l* through some point on *l*.

scalar matrix. A matrix whose diagonal elements are all equal while the nondiagonal elements are all 0. The identity matrix is an example.

scatterplot. A graph of the points representing a collection of data.

scientific notation. A shorthand way of writing very large or very small numbers. A number expressed in scientific notation is expressed as a decimal number between 1 and 10 multiplied by a power of 10 (e.g., $7000 = 7 \times 10^3$ or $0.0000019 = 1.9 \times 10^{-6}$).

similarity. In geometry, two shapes *R* and *S* are similar if there is a dilation *D* (see the definition of *dilation*) that takes *S* to a shape congruent to *R*. It follows that *R* and *S* are similar if they are congruent after one of them is expanded or shrunk.

sine. Sin(θ) is the *y*-coordinate of the point on the unit circle so that the ray connecting the point with the origin makes an angle of θ with the positive *x*-axis. When θ is an angle of a right triangle, then *sin*(θ) is the ratio of the opposite side with the hypotenuse.

square root. The square roots of *n* are all the numbers *m* so that $m^2 = n$. The square roots of 16 are 4 and -4. The square roots of -16 are 4 *i* and -4 *i*.

standard deviation. A statistic that measures the dispersion of a sample.

symmetry. A symmetry of a shape *S* in the plane or space is a rigid motion *T* that takes *S* onto itself *(T(S) = S)*. For example, reflection through a diagonal and a rotation through a right angle about the center are both symmetries of the square.

system of linear equations. Set of equations of the first degree (e.g., $x + y = 7$ and $x - y = 1$). A solution of a set of linear equations is a set of numbers *a*, *b*, *c*, . . . so that when the variables are replaced by the numbers all the equations are satisfied. For example, in the equations above, $x = 4$ and $y = 3$ is a solution.

translation. A rigid motion of the plane or space of the form *X* goes to *X* + *V* for a fixed vector *V*.

transversal. In geometry, given two or more lines in the plane a transversal is a line distinct from the original lines and intersects each of the given lines in a single point.

unit fraction. A fraction whose numerator is 1 (e.g., $1/\pi$, $1/3$, $1/x$). Every nonzero number may be written as a unit fraction since, for *n* not equal to 0, $n = 1/(1/n)$.

variable. A placeholder in algebraic expressions; for example, in $3x + y = 23$, *x* and *y* are variables.

vector. Quantity that has magnitude (length) and direction. It may be represented as a directed line segment.

zeros of a function. The points at which the value of a function is zero.